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Physics 35100 Mechanics
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Exam 1 Review

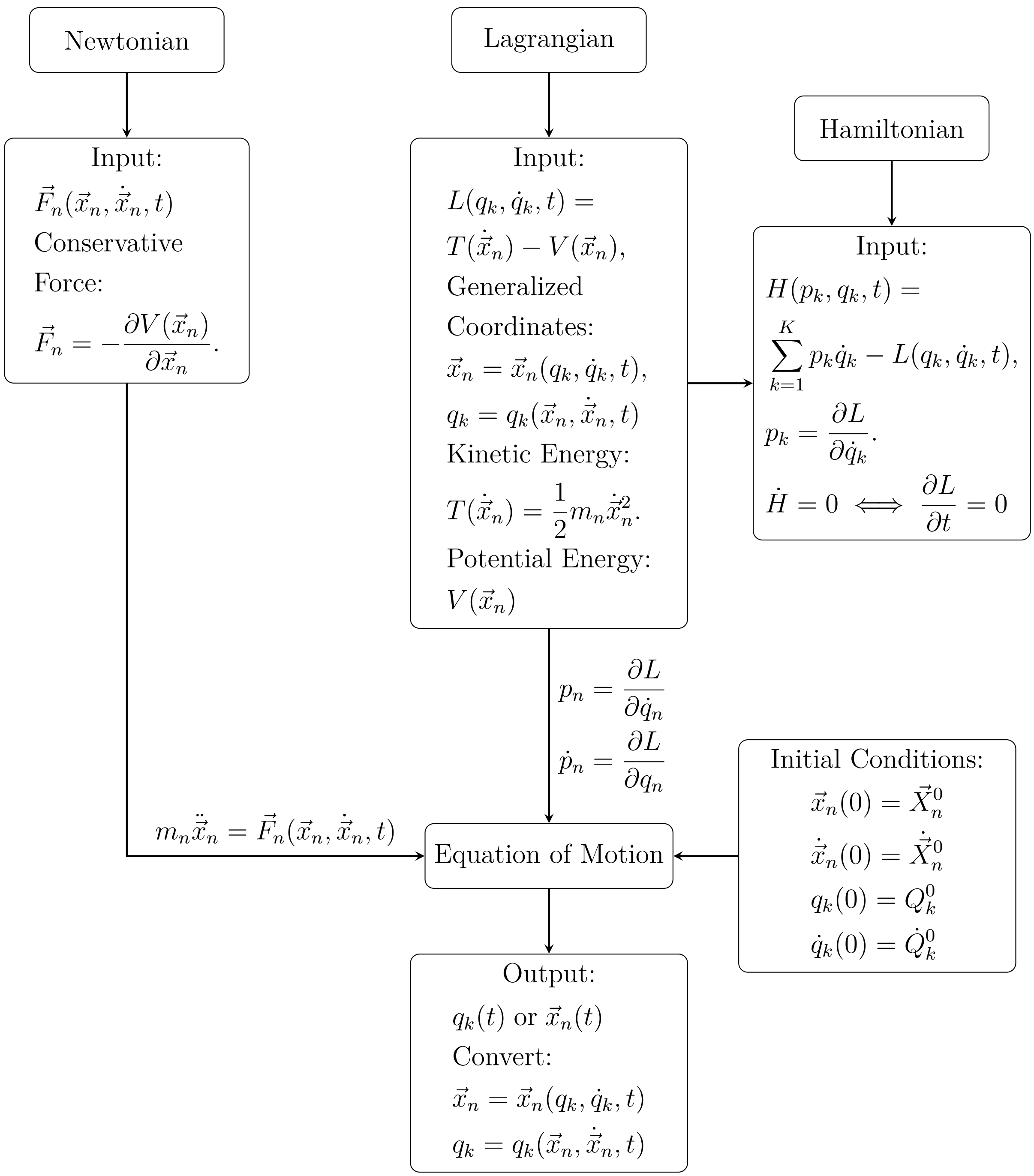

FIGURE 1. Overview of mechanics (so far).

We have studied two approaches to mechanics—Newtonian and Lagrangian. They both give the same final output, the time evolution of the coordinates, either generalized $q_k(t)$ or Cartesian $\vec{x}_n(t)$.

1. Newtonian Mechanics Review.

The most general version of Newtonian mechanics relates the kinematic description of N particles $\vec{x}_n(t)$ to the dynamic forces applied to the particles $\vec{F}_n$ in Newton's second law:

$$m_n \ddot{\vec{x}}_n = \vec{F}_n(\vec{x}_n, \dot{\vec{x}}_n, t).$$

It is often useful to break up the force into internal forces on particle n from particle k, $\vec{F}^{int}_{nk}$, and external forces, $\vec{F}^{ext}_n$, which act on an individual particle n. In that case,

$$m_n \ddot{\vec{x}}_n = \sum_{k=1}^{N} \vec{F}^{int}_{nk} + \vec{F}^{ext}_n.$$

Question 1. *Newton's Third Law*

(1) Describe the weak and strong version of Newton's third Law.

Show Answer:

(Weak form:) Forces between particles are equal in magnitude and opposite in direction:

$$\vec{F}^{int}_{nk} = -\vec{F}^{int}_{kn}$$

(Strong form:) Forces between particles are equal in magnitude and opposite in direction *and the direction is pointing from one particle center to the other*:

$$F^{int}_{nk} \hat{r}_{nk} = -F^{int}_{kn} \hat{r}_{nk},$$

where

$$\hat{r}_{nk} = \frac{\vec{r}_{nk}}{|\vec{r}_{nk}|},$$

and

$$\vec{r}_{nk} = \vec{x}_k - \vec{x}_n.$$

(2) Define the center of mass and derive the equation of motion for it using the weak form of Newton's third Law in terms of the total external force:

$$\vec{F}^{ext}_{total} = \sum_{n=1}^{N} \vec{F}^{ext}_n.$$

Show Answer:

Center of Mass,

$$\vec{X}_{cm} = \frac{\sum_{n=1}^{N} m_n \vec{x}_n}{M}$$

where

$$M = \sum_{n=1}^{N} m_n.$$

is the total mass.

$$\vec{X}_{cm} = \frac{\sum_{n=1}^{N} m_n \vec{x}_n}{M}$$
$$M\vec{X}_{cm} = \sum_{n=1}^{N} m_n \vec{x}_n$$
$$M\ddot{\vec{X}}_{cm} = \sum_{n=1}^{N} m_n \ddot{\vec{x}}_n$$
$$m_n \ddot{\vec{x}}_n = \sum_{k=1}^{N} \vec{F}^{int}_{nk} + \vec{F}^{ext}_n. \quad \text{(Newton's second law)}$$
$$M\ddot{\vec{X}}_{cm} = \sum_{n=1}^{N}\sum_{k=1}^{N} \vec{F}^{int}_{nk} + \sum_{n=1}^{N} \vec{F}^{ext}_n$$
$$= \sum_{n=1}^{N-1}\sum_{k=n+1}^{N} \vec{F}^{int}_{nk} + \vec{F}^{int}_{kn} + \sum_{n=1}^{N} \vec{F}^{ext}_n$$
$$\vec{F}^{int}_{nk} = -\vec{F}^{int}_{kn} \quad \text{(Newton's third law)}$$
$$M\ddot{\vec{X}}_{cm} = \sum_{n=1}^{N-1}\sum_{k=n+1}^{N} \vec{F}^{int}_{nk} - \vec{F}^{int}_{nk} + \sum_{n=1}^{N} \vec{F}^{ext}_n$$
$$M\ddot{\vec{X}}_{cm} = \sum_{n=1}^{N} \vec{F}^{ext}_n \equiv \vec{F}^{ext}_{total}$$

(3) Define the total angular momentum about the stationary point $\vec{x}_0$ and derive the equation of motion for it using the strong form of Newton's third Law in terms of the total external torque:

$$\vec{N}^{ext}_{total} = \sum_{n=1}^{N} \vec{r}_n \times \vec{F}^{ext}_n.$$

$\boxed{\textit{Show Answer:}}$

Total angular momentum,

$$\vec{L} = -\sum_{n=1}^{N} m_n \dot{\vec{x}}_n \times (\vec{x}_n - \vec{x}_0)$$
$$\vec{L} = -\sum_{n=1}^{N} m_n \dot{\vec{r}}_n \times \vec{r}_n$$

where

$$\vec{r}_n = \vec{x}_n - \vec{x}_0.$$

$$
\begin{aligned}
\vec{L} &= -\sum_{n=1}^{N} m_n \dot{\vec{r}}_n \times \vec{r}_n \\
\dot{\vec{L}} &= -\sum_{n=1}^{N} m_n \ddot{\vec{r}}_n \times \vec{r}_n + \dot{\vec{r}}_n \times \dot{\vec{r}}_n \\
&= -\sum_{n=1}^{N} m_n \ddot{\vec{r}}_n \times \vec{r}_n \quad (\vec{a} \times \vec{a} = 0) \\
m_n \ddot{\vec{x}}_n &= \sum_{k=1}^{N} \vec{F}_{nk}^{int} + \vec{F}_n^{ext}. \quad \text{(Newton's second law)} \\
\dot{\vec{L}} &= -\sum_{n=1}^{N} (\sum_{k=1}^{N} \vec{F}_{nk}^{int} + \vec{F}_n^{ext}) \times \vec{r}_n \\
&= -\sum_{n=1}^{N} \sum_{k=1}^{N} \vec{F}_{nk}^{int} \times \vec{r}_n - \sum_{n=1}^{N} \vec{F}_n^{ext} \times \vec{r}_n \\
&= -\sum_{n=1}^{N-1} \sum_{k=n+1}^{N} (\vec{F}_{nk}^{int} \times \vec{r}_n + \vec{F}_{kn}^{int} \times \vec{r}_k) - \sum_{n=1}^{N} \vec{F}_n^{ext} \times \vec{r}_n \\
F_{nk}^{int} \hat{r}_{nk} &= -F_{kn}^{int} \hat{r}_{nk} \quad \text{(Newton's third law)} \\
\dot{\vec{L}} &= -\sum_{n=1}^{N-1} \sum_{k=n+1}^{N} F_{nk}^{int} (\hat{r}_{nk} \times \vec{r}_n - \hat{r}_{nk} \times \vec{r}_k) - \sum_{n=1}^{N} \vec{F}_n^{ext} \times \vec{r}_n \\
&= -\sum_{n=1}^{N-1} \sum_{k=n+1}^{N} F_{nk}^{int} \hat{r}_{nk} \times (\vec{r}_n - \vec{r}_k) - \sum_{n=1}^{N} \vec{F}_n^{ext} \times \vec{r}_n \\
\vec{r}_{nk} &= \vec{x}_k - \vec{x}_n = \vec{x}_k - \vec{x}_0 - (\vec{x}_n - \vec{x}_0) = \vec{r}_k - \vec{r}_n \\
\dot{\vec{L}} &= \sum_{n=1}^{N-1} \sum_{k=n+1}^{N} F_{nk}^{int} \hat{r}_{nk} \times \vec{r}_{nk} - \sum_{n=1}^{N} \vec{F}_n^{ext} \times \vec{r}_n \\
\hat{a} \times \vec{a} &= 0 \; \forall \vec{a} \\
\dot{\vec{L}} &= -\sum_{n=1}^{N} \vec{F}_n^{ext} \times \vec{r}_n = \sum_{n=1}^{N} \vec{r}_n \times \vec{F}_n^{ext} \equiv \vec{N}_{total}^{ext}
\end{aligned}
$$

Question 2. *One particle, force depends only on time.* A particle of mass $m = 3\,\text{kg}$ initially $t = 0\,s$ at position $\vec{x}(0) = (-3, 2, -3)\,m$ and initial velocity $\dot{\vec{x}}(0) = (1, -2, 3)\,m/s$ is acted on by a force $\vec{F}(t) = (3t, -2, 3\sin t)\,N$.

(1) Find a general expression for the position of the particle $\vec{x}(t)$ as a function of time t.
Show Answer:

$$
\begin{aligned}
m\ddot{\vec{x}}(t) &= \vec{F}(t) = (3t, -2, 3\sin t) \\
m\dot{\vec{x}}(t) &= (3/2t^2, -2t, -3\cos t) + m\vec{V}_0 \\
\dot{\vec{x}}(0) &= (0, 0, -3)/m + \vec{V}_0 = (1, -2, 3) \\
\vec{V}_0 &= (1, -2, 3) - (0, 0, -1) = (1, -2, 4) \\
m\vec{x}(t) &= (1/2t^3, -t^2, -3\sin t) + m\vec{V}_0 t + m\vec{X}_0 \\
\vec{x}(0) &= (0, 0, 0) + \vec{X}_0 = (-3, 2, -3) \\
\vec{X}_0 &= (-3, 2, -3) \\
\vec{x}(t) &= (1/2t^3, -t^2, -3\sin t)/m + \vec{V}_0 t + \vec{X}_0 \\
&\boxed{\vec{x}(t) = (1/6t^3 + t - 3, -1/3t^2 - 2t + 2, 4t - 3 - \sin t)\,m}
\end{aligned}
$$

Check Answer:

$$
\begin{aligned}
\vec{x}(0) &= (-3, 2, -3)\,m \quad \checkmark \\
\dot{\vec{x}}(t) &= (1/2t^2 + 1, -2/3t - 2, 4 - \cos t)\,m/s \\
\dot{\vec{x}}(0) &= (1, -2, 3)\,m/s \quad \checkmark \\
\ddot{\vec{x}}(t) &= (t, -2/3, \sin t)\,m/s^2 \\
m\ddot{\vec{x}}(t) &= \vec{F}(t) = (3t^3, -2, 3\sin t)\,N \quad \checkmark
\end{aligned}
$$

(2) What time t is the $\hat{z} = (0, 0, 1)$ velocity $v_z = \dot{\vec{x}} \cdot \hat{z}$ maximum in the range of $0\,s \le t \le 6\,s$? What is the acceleration vector at that point?
Show Answer:

$$
\begin{aligned}
v_z &= \dot{\vec{x}}(t) \cdot \hat{z} = (1/2t^2 + 1, -2/3t - 2, 4 - \cos t) \cdot (0, 0, 1)\,m/s = 4 - \cos t \\
\dot{v}_z &= \sin t = 0 \\
t &= n\pi, n \in \mathbb{Z} \\
t &= \{0, \pi\}, (0\,s \le t \le 6) \\
\ddot{v}_z &= \cos t \\
\cos 0 > 0&, \cos\pi < 0 \checkmark \\
t &= \pi\,s \approx 3.14\,s \\
\ddot{\vec{x}}(t) &= (t, -2/3, \sin t)\,m/s^2 \\
\ddot{\vec{x}}(\pi) &= (\pi, -2/3, 0)\,m/s^2 \approx (3.14, -2/3, 0)\,m/s^2
\end{aligned}
$$

(3) Show that the Impulse-Momentum Theorem is true for the $\hat{z}$ component of the motion during the interval $0 \leq t \leq \pi$?

$\boxed{\textit{Show Answer:}}$

$$
\begin{aligned}
J_z = \vec{J} \cdot \hat{z} &= \int_0^\pi \vec{F}(t) \cdot \hat{z}\, dt = \Delta\vec{p} \cdot \hat{z} \\
&= \int_0^\pi 3 \sin t\, dt = m(v_z(\pi) - v_z(0)) \\
&= -3 \cos t|_0^\pi = 3(4 - \cos\pi - 4 + \cos 0) \\
&= -3(-1 - 1) = 3(4 + 1 - 4 + 1) \\
J_z &= 6 = 6
\end{aligned}
$$

Question 3. *One particle, force depends only space.* All known real forces are conservative. A conservative force can be derived from a potential energy V. Consider a mass m at the end of mass-less spring with spring constant K and rest length L_0 hanging from a rigid immovable beam under gravity. Assume the motion is one dimensional in the direction of gravity.

(1) Measuring the position y of the mass from beam with y increasing downward and the zero of gravitational potential at $y = 0$, what is the potential energy of the mass-spring system?

$\boxed{\textit{Show Answer:}}$ The potential from the spring is:

$$V_{spring}(y) = \frac{1}{2}K(y - L_0)^2.$$

The potential from gravity is:

$$V_{gravity}(y) = -mgy$$

Then the total potential

$$V(y) = V_{spring}(y) + V_{gravity}(y) = \frac{1}{2}K(y - L0)^2 - mgy$$

(2) Find the equation of motion for the mass.

$\boxed{\textit{Show Answer:}}$

$$
\begin{aligned}
m\ddot{y} &= -\frac{\partial V}{\partial y} \\
m\ddot{y} &= -K(y - L_0) + mg \\
\ddot{y} &= -\frac{K}{m}(y - L_0) + g
\end{aligned}
$$

(3) Non-dimensionalize the equation using $[L] = L_0$, $[T] = \sqrt{\frac{m}{K}}$, and $[M] = m$.

$\boxed{\textit{Show Answer:}}$

$$
\begin{aligned}
t &= \sqrt{\frac{m}{K}}\tau \\
\tau &= \sqrt{\frac{K}{m}}t \\
\frac{d\tau}{dt} &= \sqrt{\frac{K}{m}} \\
y(t) &= L_0 u(\tau) \\
\dot{y}(t) &= \frac{dy(t)}{dt} \\
&= \frac{d(L_0 u(\tau))}{dt} \\
&= L_0 \frac{du(\tau)}{d\tau}\frac{d\tau}{dt} \\
&= L_0 \frac{du(\tau)}{d\tau}\sqrt{\frac{K}{m}} \\
&= L_0 \dot{u}(\tau)\sqrt{\frac{K}{m}} \\
\dot{y}(t) &= L_0\sqrt{\frac{K}{m}}\dot{u}(\tau) \\
\ddot{y}(t) &= L_0\frac{K}{m}\ddot{u}(\tau) \\
\ddot{y}(t) &= -\frac{K}{m}(y(t) - L_0) + g \\
L_0\frac{K}{m}\ddot{u}(\tau) &= -\frac{K}{m}(L_0 u(\tau) - L_0) + g \\
&= -L_0\frac{K}{m}(u(\tau) - 1) + g \\
\ddot{u}(\tau) &= -u(\tau) - 1 + \frac{mg}{KL_0} \\
\ddot{u}(\tau) &= -u(\tau) + \Gamma - 1,
\end{aligned}
$$

where

$$\Gamma = \frac{mg}{KL_0}.$$

(4) Solve the dimensionless equation and then convert back to dimensional form, using initial conditions $y(0) = Y_0$ and $\dot{y}(0) = V_0$.

Show Answer:

Change of varible to simplify:

$$\begin{aligned}
\ddot{u}(\tau) &= -u(\tau) + \Gamma - 1, \\
\ddot{u}(\tau) &= -(u(\tau) - \Gamma + 1), \\
q(\tau) &= u(\tau) - \Gamma + 1 \\
\ddot{q}(\tau) &= \ddot{u}(\tau) = -q(\tau)
\end{aligned}$$

Assume:

$$\begin{aligned}
q(\tau) &= A \sin \omega\tau + B \cos \omega\tau \\
\dot{q}(\tau) &= A\omega \cos \omega\tau - B\omega \sin \omega\tau \\
\ddot{q}(\tau) &= -q = -A\omega^2 \sin \omega\tau - B\omega^2 \cos \omega\tau \\
-A \sin \omega\tau - B \cos \omega\tau &= -A\omega^2 \sin \omega\tau - B\omega^2 \cos \omega\tau \\
0 &= A(1 - \omega^2) \sin \omega\tau + B(1 - \omega^2) \cos \omega\tau \\
\omega^2 &= 1 \\
\omega &= \pm 1 \\
q(\tau) &= \pm A \sin \tau + B \cos \tau.
\end{aligned}$$

From the general solution apply the initial conditions:

$$\begin{aligned}
q(\tau) &= u(\tau) - \Gamma + 1 = \frac{y(t)}{L_0} - \Gamma + 1 \\
q(0) &= \pm A \sin 0 + B \cos 0 = \frac{Y_0}{L_0} - \Gamma + 1 \\
B &= \frac{Y_0}{L_0} - \Gamma + 1 \\
\dot{q}(\tau) &= \pm A \cos \tau - B \sin \tau \\
\dot{y}(t) &= L_0 \sqrt{\frac{K}{m}} \dot{u}(\tau) \\
\dot{q}(\tau) &= \dot{u}(\tau) = \sqrt{\frac{m}{K}} \frac{\dot{y}(t)}{L_0} \\
\dot{q}(0) &= \pm A \cos 0 - B \sin 0 = \sqrt{\frac{m}{K}} \frac{V_0}{L_0} \\
\pm A &= \sqrt{\frac{mV_0^2}{KL_0^2}} \\
q(\tau) &= \sqrt{\frac{mV_0^2}{KL_0^2}} \sin \tau + (\frac{Y_0}{L_0} - \Gamma + 1) \cos \tau
\end{aligned}$$

Now convert back to $y(t)$:

$$\begin{aligned}
y(t) &= L_0(q(\tau) + \Gamma - 1) \\
&= L_0\left(\sqrt{\frac{mV_0^2}{KL_0^2}}\sin\tau + \left(\frac{Y_0}{L_0} - \Gamma + 1\right)\cos\tau + \Gamma - 1\right) \\
&= \sqrt{\frac{m}{K}}V_0\sin\tau(t) + (Y_0 - L_0(\Gamma - 1))\cos\tau(t) + L_0(\Gamma - 1) \\
y(t) &= L_1 + (Y_0 - L_1)\cos\omega_0 t + \frac{V_0}{\omega_0}\sin\omega_0 t
\end{aligned}$$

where

$$\omega_0 = \sqrt{\frac{K}{m}},$$

and

$$L_1 = L_0(1 - \Gamma) = L_0 - \frac{g}{\omega_0^2}.$$

Question 4. *One particle, force depends position and velocity* A damped harmonic oscillator has equation of motion:

$$m\ddot{x} = -Kx - B\dot{x}$$

(1) Find a general solution $x(t)$.

$\boxed{\textit{Show Answer:}}$

$$\begin{aligned}
m\ddot{x} &= -Kx - B\dot{x} \\
\ddot{x} &= -\omega_0^2 x - \beta\dot{x} \\
\text{where, } \omega_0 &= \sqrt{\frac{K}{m}} \text{ and } \beta = \frac{B}{m} \\
\text{Guess: } x(t) &= A\exp\Omega t \\
\dot{x}(t) &= A\Omega\exp\Omega t \\
\ddot{x}(t) &= A\Omega^2\exp\Omega t \\
A\Omega^2\exp\Omega t &= -A\omega_0^2\exp\Omega t - A\beta\Omega\exp\Omega t \\
0 &= A(\Omega^2 + \omega_0^2 + \beta\Omega)\exp\Omega t \\
0 &= \Omega^2 + \omega_0^2 + \beta\Omega \\
\Omega &= \frac{1}{2}(-\beta \pm \sqrt{\beta^2 - 4\omega_0^2}) \\
x(t) &= A\exp\left[\frac{1}{2}(-\beta + \sqrt{\beta^2 - 4\omega_0^2})t\right] + B\exp\left[\frac{1}{2}(-\beta - \sqrt{\beta^2 - 4\omega_0^2})t\right]
\end{aligned}$$

(2) Assuming $B > 0$ what kind of solution will you get if

(a) $(B/m)^2 > 4K/m$.

(b) $(B/m)^2 > 4K/m$.

(c) $(B/m)^2 = 4K/m$.

$\boxed{\textit{Show Answer:}}$

$$
\begin{aligned}
(B/m)^2 &> 4K/m \\
\beta^2 &> 4\omega_0^2 \\
-\beta \pm \sqrt{\beta^2 - 4\omega_0^2} &\in \mathbb{R}
\end{aligned}
$$

The solution will exponentially decay.

$$
\begin{aligned}
(B/m)^2 &< 4K/m \\
\beta^2 &< 4\omega_0^2 \\
-\beta \pm \sqrt{\beta^2 - 4\omega_0^2} &\in \mathbb{C}
\end{aligned}
$$

The solution will exponentially decay and oscillate because of the imaginary part.

$$
\begin{aligned}
(B/m)^2 &= 4K/m \\
\beta^2 &= 4\omega_0^2 \\
-\beta \pm \sqrt{\beta^2 - 4\omega_0^2} &= -\beta
\end{aligned}
$$

The solution will exponentially decay like $\exp(-\beta/2)$.

4.1. Lagrangian Mechanics Review.

Lagrangian mechanics is equivalent to Newtonian mechanics, but it has two major advantages. 1) It is relatively easy to treat generalized coordinates. This make changing coordinates easier and allows the elimination of constraints very simple. 2) It is easy to include continuous symmetries and each continuous symmetry leads to a conserved quantity. The three most important symmetries space-translation, rotation, and time translation lead to conservation of momentum, angular momentum, Energy. The most general version of Lagrangian mechanics has a an input the Lagrangian function $L(t) = L(q_k(t), \dot{q}_k(t), t)$. The q_k are K generalized coordinates. The dynamics are determined from requiring that the action,

$$S = \int_{t_1}^{t_2} L(t)dt$$

is stationary (i.e., $\delta S = 0$). Using variational calculus the stationary requirement lead to K Euler-Lagrange equations:

$$\ddot{p}_k = \frac{\partial L}{\partial q_k},$$

where

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

is the generalized momentum.

To calculate the Lagrangian we use the definition in Cartesian coordinate $\vec{x}_n(t)$, such that:

$$L(\vec{x}_n(t), \dot{\vec{x}}_n(t)) = T(\dot{\vec{x}}_n(t)) - V(\vec{x}_n(t)),$$

where

$$T(\dot{\vec{x}}_n(t)) = \frac{1}{2} m_n \dot{\vec{x}}_n^2(t),$$

is the kinetic energy and $V(\vec{x}_n(t))$ is the potential energy of the system. In Cartesian coordinates there are N particles in D dimensions. The vector part of the coordinates $\vec{x}_n(t)$ represents the D dimension of the space that the particles are in and the subscript represents the number of the nth particle from a total of N particles. The Lagrangian depends on a total of DN coordinates and DN velocities $\dot{\vec{x}}_n(t)$. To convert to generalized coordinates a change of variables is used:

$$\vec{x}_n = \vec{x}_n(q_k(t), \dot{q}_k(t), t)$$

along with an inverse transform:

$$q_k = q_k(\vec{x}_n(t), \dot{\vec{x}}_n(t), t).$$

In the transformation, the number of coordinates (degrees of freedom) K in the generalize coordinates must equal the number of degrees of freedom in the Cartesian coordinates DN. However, if the $q_k(t)$ are chosen to include constrained variables then they can be easily eliminated from the final equations of motion.

Question 5. *Equivalence of Lagrangian and Newtonian mechanics*

(1) A general Lagrangian is given by

$$L(\vec{x}_n(t), \dot{\vec{x}}_n(t)) = \frac{1}{2} m_n \dot{\vec{x}}_n^2(t) - V(\vec{x}_n(t)).$$

Show that the Euler-Lagrange equation leads to the Newtonian equations of motion for particles in a conservative potential $V(\vec{x}_n(t))$.

Show Answer:

$$\begin{aligned}
L(\vec{x}_n(t), \dot{\vec{x}}_n(t)) &= \frac{1}{2} m_n \dot{\vec{x}}_n^2(t) - V(\vec{x}_n(t)) \\
\vec{p}_n &= \frac{\partial L}{\partial \dot{\vec{x}}} \\
&= m_n \dot{\vec{x}}_n(t) \\
\dot{\vec{p}}_n &= -\frac{\partial V}{\partial \vec{x}} \\
m_n \ddot{\vec{x}}_n(t) &= -\frac{\partial V}{\partial \vec{x}} = \vec{F}_n
\end{aligned}$$

Question 6. *Double Pendulum* A double pendulum consists of a mass m_1 at postion $\vec{x}_1$ connected by a rigid mass-less rod of length L_1 to a rigid immovable beam. A second rigid mass-less beam of length L_2 connects m_1 to a second mass m_2 at position $\vec{x}_2$. The masses are confined to the x-y plane with gravity pointing downward in the $-\hat{y}$ direction.

(1) How many degrees of freedom are needed to represent the positions $\vec{x}_1$ and $\vec{x}_2$ in Cartesian coordinates?

Show Answer:

4. Each vector requires 2 dimensions and there are 2 positions so a total of $2 * 2 = 4$ degrees of freedom are needed. Explicitly (x_1, y_1, x_2, y_2) define the two vectors $\vec{x}_1 = (x_1, y_1)$ and $\vec{x}_2 = (x_2, y_2)$.

(2) Constraints:
  (a) How many constrains of the form $f(\vec{x}_n) = 0$ are there?
  (b) Express them in the form $f(\vec{x}_n) = 0$.

(c) Are there any other constraints?

Show Answer:

(a) 2. The distance between each mass.
(b) $\vec{x}_1 \cdot \vec{x}_1 - L_1^2 = 0$ and $(\vec{x}_2 - \vec{x}_1) \cdot (\vec{x}_2 - \vec{x}_1) - L_2^2 = 0 = d_{12}^2 - L_2^2$
(c) No. The beam position represents a constraint, but it is not included in the Cartesian coordinates. The constraint to move in a plane $z_1 = 0$ and $z_2 = 0$ is also not in the Cartesian coordinate description.

(3) Write out a coordinate transformation $\vec{x}_n = \vec{x}_n(q_k)$, which will include two generalized coordinates that have zero time derivatives. This is always possible with constraints of the form: $f(\vec{x}_n) = 0$ since

$$\frac{df(\vec{x}_n)}{dt} = \frac{df(\vec{x}_n)}{d\vec{x}_n}\frac{d\vec{x}_n}{dt} = \frac{df(\vec{x}_n)}{d\vec{x}_n}\dot{\vec{x}}_n = 0,$$

and

$$\frac{df(\vec{x}_n)}{d\vec{x}_n} \neq 0$$

because there is a constraint and therefore $\dot{\vec{x}}_n = 0$.

Show Answer:

$$\begin{aligned}
\vec{x}_n &= \vec{x}_n(q_k) = \vec{x}_n(L_1, \theta_1, L_2, \theta_2) \\
\vec{x}_1 &= \vec{l}_1 \\
\vec{x}_2 &= \vec{l}_1 + \vec{l}_2 \\
\vec{l}_1 &= (L_1 \cos\theta_1, L_1 \sin\theta_1) \\
\vec{l}_2 &= (L_2 \cos\theta_2, L_2 \sin\theta_2)
\end{aligned}$$

(4) What is the inverse transform $q_k = q_k(\vec{x}_n)$?

Show Answer:

$$\begin{aligned}
q_k &= q_k(\vec{x}_n) = q_k(\vec{x}_1, \vec{x}_1) \\
L_1 &= |\vec{x}_1| \\
L_2 &= |\vec{x}_2 - \vec{x}_1| \\
\theta_1 &= \arctan\left(\frac{y_1}{x_1}\right) \\
\theta_2 &= \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)
\end{aligned}$$

(5) What is the Lagrangian in Cartesian coordinate, $\vec{x}_n$?

Show Answer:

$$\begin{aligned}
L(\vec{x}_n, \dot{\vec{x}}_n) &= T(\dot{\vec{x}}_n) - V(\vec{x}_n) \\
&= \frac{1}{2}\left(m_1\dot{\vec{x}}_1^2 + m_2\dot{\vec{x}}_2^2\right) - V(\vec{x}_n)
\end{aligned}$$

The potential $V(\vec{x}_n)$ is due to gravity so:

$$\begin{aligned} V(\vec{x}_n) &= m_1 g(\vec{x}_1 \cdot \hat{y}) + m_2 g(\vec{x}_2 \cdot \hat{y}) \\ &= m_1 g y_1 + m_2 g y_2 \end{aligned}$$

and

$$L(\vec{x}_n, \dot{\vec{x}}_n) = \frac{1}{2}\left(m_1 \dot{\vec{x}}_1^2 + m_2 \dot{\vec{x}}_2^2\right) - m_1 g y_1 - m_2 g y_2$$

(6) What is the Lagrangian in generalized coordinate, q_k?

$\boxed{\textit{Show Answer:}}$

$$\begin{aligned} L(\vec{x}_n, \dot{\vec{x}}_n) &= \frac{1}{2}\left(m_1 \dot{\vec{x}}_1^2 + m_2 \dot{\vec{x}}_2^2\right) - m_1 g y_1 - m_2 g y_2 \\ \dot{\vec{x}}_1 &= \dot{\vec{l}}_1 = \frac{d}{dt}(L_1 \cos\theta_1, L_1 \sin\theta_1) \\ &= (-L_1\dot{\theta}_1 \sin\theta_1, L_1\dot{\theta}_1 \cos\theta_1) \\ \dot{\vec{x}}_1^2 &= \dot{\vec{l}}_1^2 = L_1^2\dot{\theta}_1^2(\sin^2\theta_1 + \cos^2\theta_1) = L_1^2\dot{\theta}_1^2 \\ \dot{\vec{l}}_2 &= \frac{d}{dt}(L_2\cos\theta_2, L_2\sin\theta_2) \\ &= (-L_2\dot{\theta}_2\sin\theta_2, L_2\dot{\theta}_2\cos\theta_2) \\ \dot{\vec{l}}_2^2 &= L_2^2\dot{\theta}_2^2 \\ \dot{\vec{x}}_2 &= \dot{\vec{l}}_2 - \dot{\vec{l}}_1 \\ \dot{\vec{x}}_2^2 &= \dot{\vec{l}}_2^2 - 2\dot{\vec{l}}_2 \cdot \dot{\vec{l}}_1 + \dot{\vec{l}}_1^2 \\ \dot{\vec{l}}_2 \cdot \dot{\vec{l}}_1 &= L_1 L_2 \dot{\theta}_1\dot{\theta}_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) \\ &= L_1 L_2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 - \theta_2 \\ \dot{\vec{x}}_2^2 &= L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \\ L &= \frac{1}{2}\left[(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2(L_2^2\dot{\theta}_2^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] - m_1 g L_1\sin\theta_1 - m_2 g L_2 \sin\theta_2 \end{aligned}$$

(7) Find the generalized momenta for the non-constrained variables?

$\boxed{\textit{Show Answer:}}$

$$\begin{aligned} L &= \frac{1}{2}\left[(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2(L_2^2\dot{\theta}_2^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] - m_1 g L_1\sin\theta_1 - m_2 g L_2 \sin\theta_2 \\ p_{\theta_1} &= \frac{\partial L}{\partial\dot{\theta}_1} = (m_1+m_2)L_1^2\dot{\theta}_1 - m_2 L_1 L_2\dot{\theta}_2\cos(\theta_1-\theta_2) \\ p_{\theta_2} &= \frac{\partial L}{\partial\dot{\theta}_2} = m_2 L_2^2\dot{\theta}_2 - m_2 L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \end{aligned}$$

(8) Find the Hamiltonian?

$\boxed{\textit{Show Answer:}}$

$$
\begin{aligned}
H &= \sum_{k=1}^{K} p_k \dot{q}_k - L \\
&= p_{\theta_1}\dot{\theta}_1 + p_{\theta_2}\dot{\theta}_2 \\
&= (m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2(L_2^2\dot{\theta}_2^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) - L \\
H &= \frac{1}{2}\left[(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2(L_2^2\dot{\theta}_2^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + m_1gL_1\sin\theta_1 + m_2gL_2\sin\theta_2
\end{aligned}
$$

(9) Use the Euler-Lagrange equations to find the equations of motion for the non-constrained variables?

$\boxed{\textit{Show Answer:}}$

$$
\begin{aligned}
L &= \frac{1}{2}\left[(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2(L_2^2\dot{\theta}_2^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] - m_1gL_1\sin\theta_1 - m_2gL_2\sin\theta_2 \\
\dot{p}_{\theta_1} &= \frac{\partial L}{\partial \theta_1} = m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_1gL_1\cos\theta_1 \\
\dot{p}_{\theta_2} &= \frac{\partial L}{\partial \theta_2} = -m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2gL_2\cos\theta_2
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{p}_{\theta_1} &= (m_1+m_2)L_1^2\ddot{\theta}_1 - m_2L_1L_2(\ddot{\theta}_2\cos(\theta_1-\theta_2) - \dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2)) \\
\dot{p}_{\theta_2} &= m_2L_2^2\ddot{\theta}_2 - m_2L_1L_2(\ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2))
\end{aligned}
$$